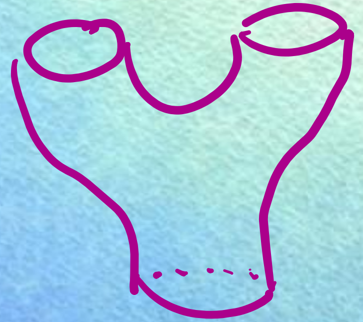
Renee Hoekzema, Oxford

Cutting and pasting in the 21st century

LMS Women in Mathematics Day

11 May 2022

Topology



pure topology : what shapes exist ? how can we describe them ?

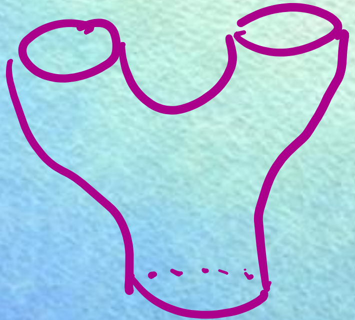
algebraic topology

applied topology : shapes in the world !

data :

knots in DNA

Topology



pure topology:

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Cutting and pasting

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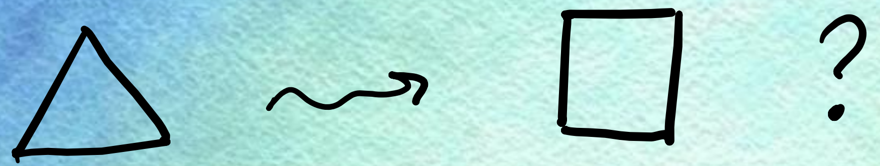
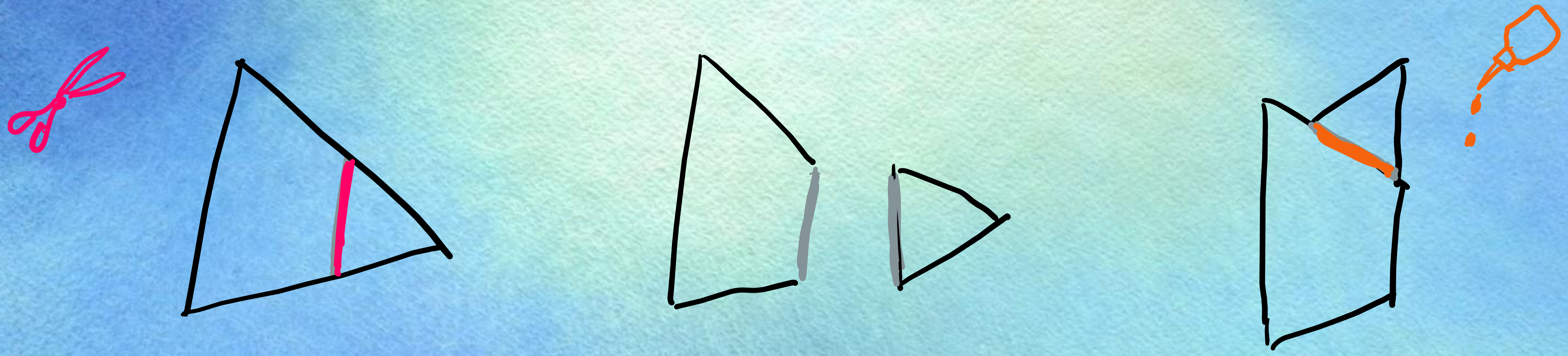
Scissors congruence

Classical question: Given two polygons P, Q, can I cut P into pieces and reassemble them to form Q?





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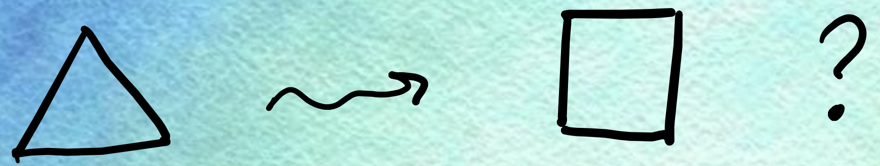
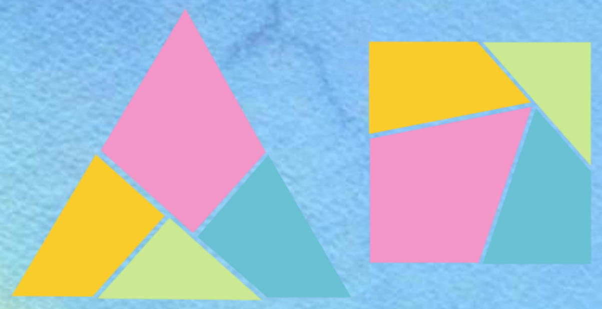
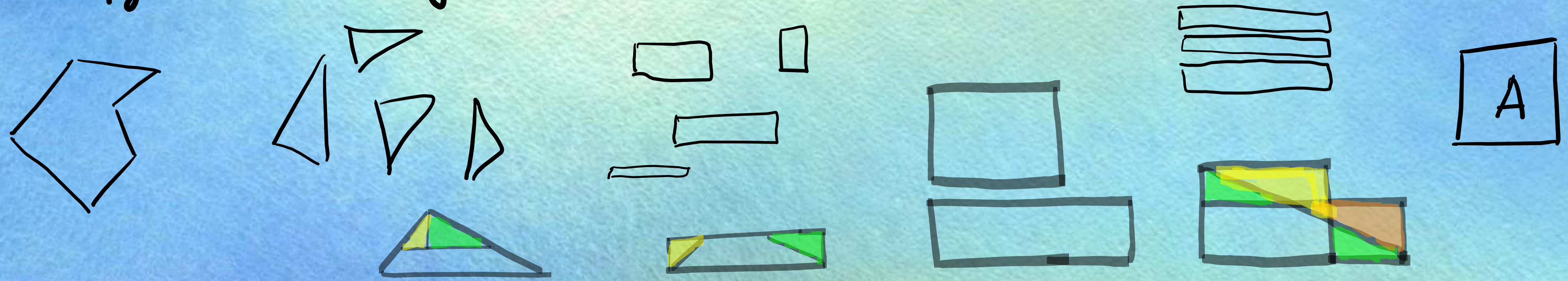




Wallace - Bolyai - Gerwien Theorem (1807): Yes, if $\text{area}(P) = \text{area}(Q)$.

Proof: polygon $\Rightarrow$ triangles $\Rightarrow$ rectangles $\Rightarrow$ rectangles of give width $\Rightarrow$ square

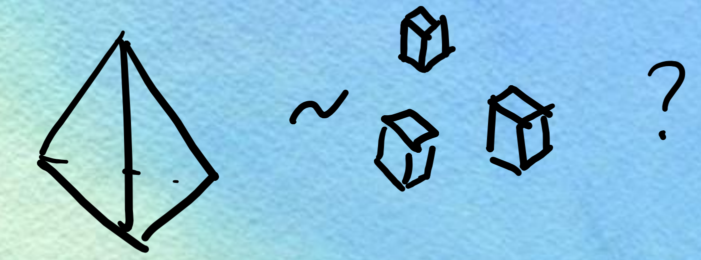


Hilbert's 3^{rd} problem: How about dim 3?

Given polyhedra P, Q of equal volume, are they equidecomposable?

In particular, is there a finite cut-and-paste argument to show the volume of a pyramid is

$$\frac{\text{base area} \times \text{height}}{3} \;?$$



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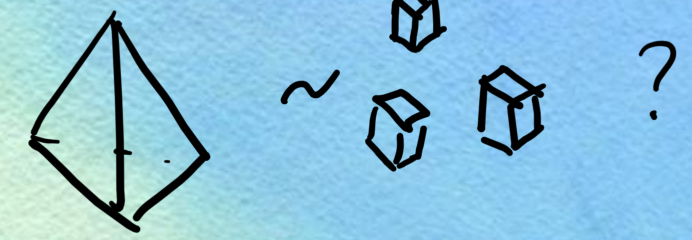
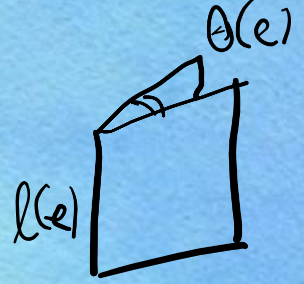


Dehn (1901): **No**

- If we cut an edge across, the sum of the lengths is preserved
- If we cut along an edge, the sum of the angles is preserved
- If we create a new edge, angles add up to 2π.



$\rightsquigarrow$ Dehn invariant is preserved

$$D(P) = \sum_e \ell(e) \otimes (\theta(e) + \pi\mathbb{Z}) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$$

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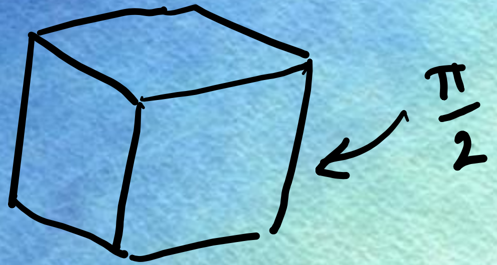

$$D(\text{cube}) =$$

$$12 \cdot \left(\ell_c \otimes \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)\right)$$

$$= 6 \cdot (\ell_c \otimes (\pi + \pi\mathbb{Z}))$$

$$= 0$$

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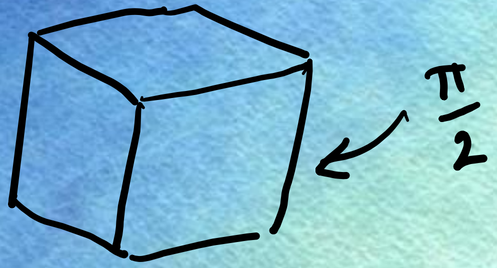

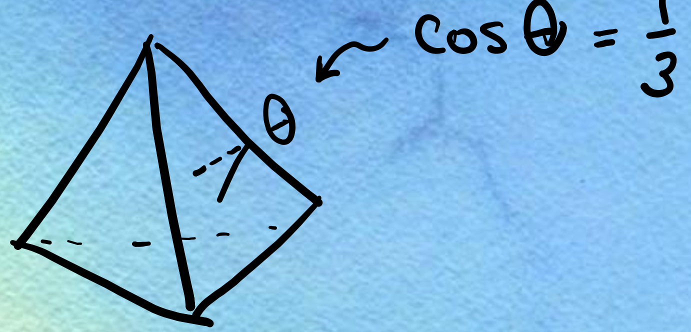

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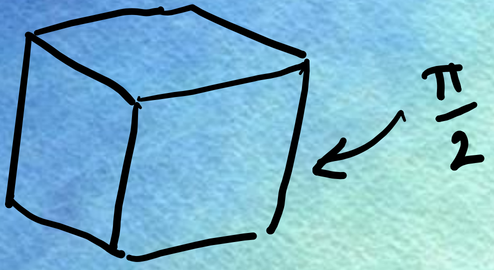
$$\arccos\left(\tfrac{1}{3}\right) \notin \mathbb{Q}.$$

$$D(\text{tetrahedron}) =$$

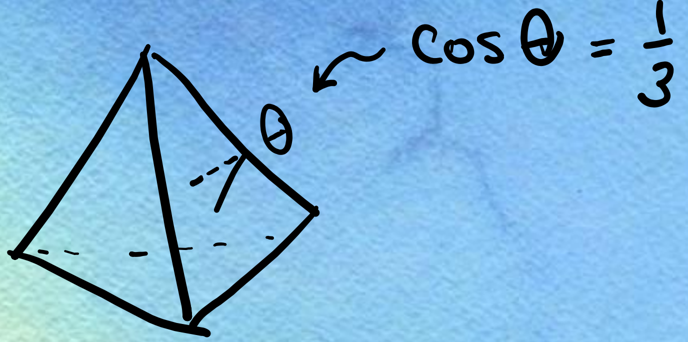
$$6 \cdot \left(\ell_T \otimes (\theta + \mathbb{Z}\pi)\right)$$

$$\neq 0.$$

$$D(P) = \sum_e \ell(e) \otimes (\theta(e) + \pi\mathbb{Z}) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$$



$\not\sim$



$D = 0$

$D \neq 0$

<u>Sydler (1965)</u>:

polyhedra are scissors congruent $\Longleftrightarrow$ same volume + Dehn invariant

How about dimension 4? 5?

Dehn invariant $\rightsquigarrow$ higher Dehn invariants

dim 4: 4-volume + Dehn invariant are only invariants.
[Jessen 1972]

dim $\geq$ 5: Open problem.

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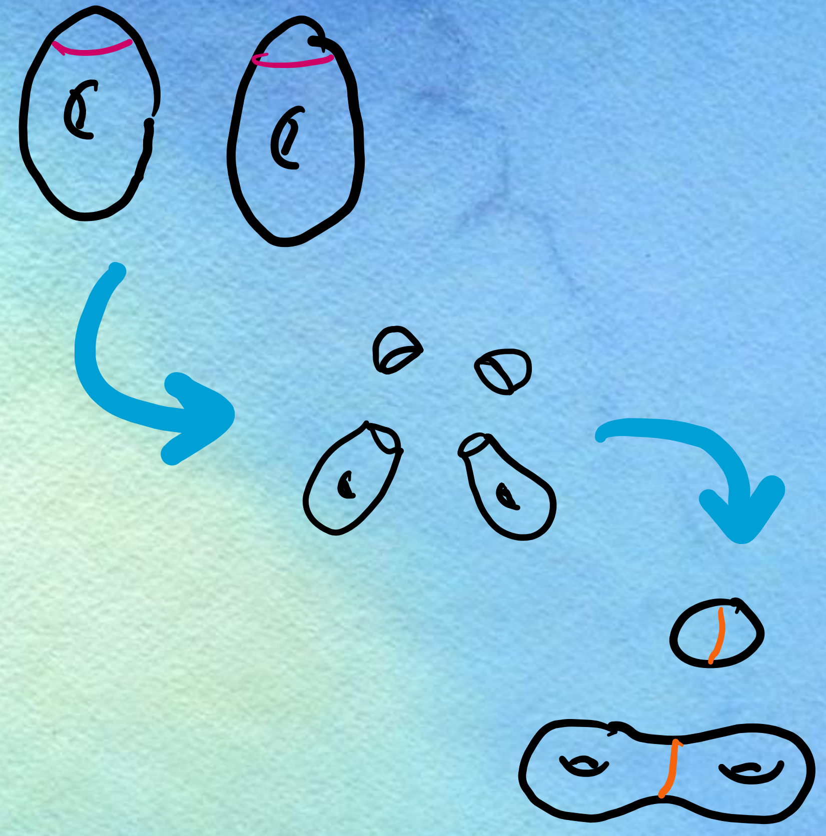
dim $\geq$ 5: Open problem.

Modern approach:
Use K-theory machinery
Inna Zakharevich & Jonathan Campbell

Our aim:
Build modern tools for a different kind of cutting & pasting

Different scissors congruence: cut-and-paste invariants of manifolds

- Take a manifold
  (smooth, closed, oriented)
- Cut along a codim 1 submanifold
- Paste back along the boundary



Def: M and M' are SK-equivalent ("schneiden und kleben") if

$$M = M_1 \cup_{\varphi} M_2 \quad \text{ad} \quad M' = M_1 \cup_{\psi} M_2$$

What are cut and paste invariants of manifolds?

Def: $SK_n :=$ n-manifolds up to SK equivalence

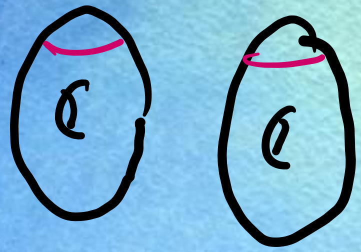
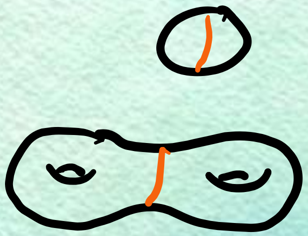
abelian group with $M + N = M \sqcup N$

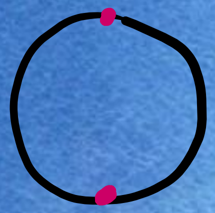
Example:

$\rightsquigarrow SK_1 = 0$

 $\sim_{SK}$  $\rightsquigarrow$ $Sk_1 = 0$

$\underline{Sk_2}$:  $\sim_{SK}$ 

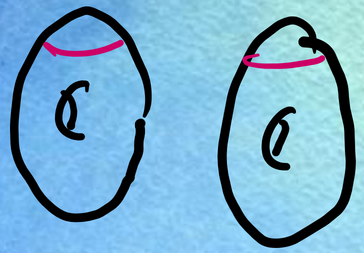
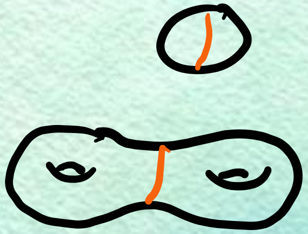
 $\sim_{SK}$ 

 $\sim_{sk}$  $\rightsquigarrow \quad Sk_1 = 0$

$\underline{Sk_2}$:  $\sim_{sk}$ 

$\chi = 0 \qquad \chi = 0 + 0$

$$Sk_2 \xrightarrow[\chi/2]{} \mathbb{Z}$$

 $\sim_{sk}$ 

$\chi = 0 + 0 \qquad \chi = 2 + -2$

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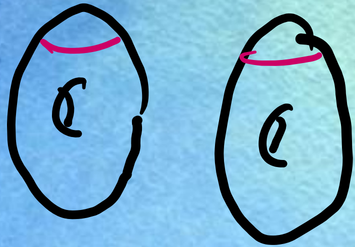
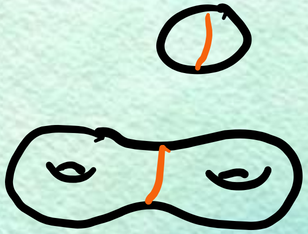
 $\sim_{SK}$ 

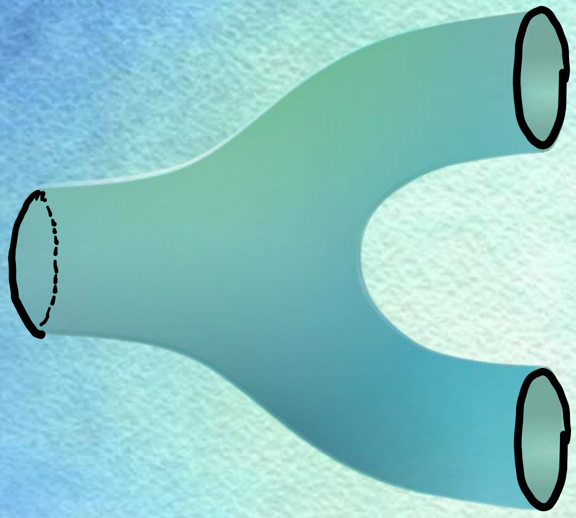
$\chi = 0 + 0$ $\qquad$ $\chi = 2 + -2$

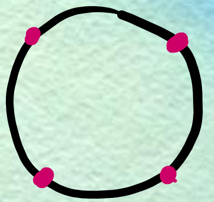
Karras, Kreck, Neumann, Ossa '73:

χ and σ are the only SK invariants

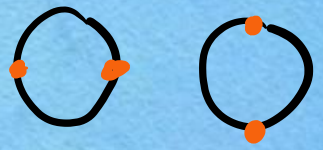
Another important equivalence of manifolds:

cobordism



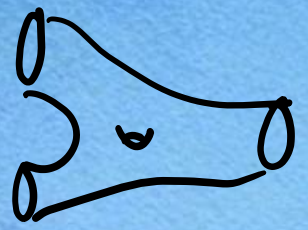


$\sim_{SK}$



How are SK groups related to cobordism invariants?

Ω_n = n-manifolds up to cobordism



$M + N = M \amalg N$

Turns out: there is no map $SK_n \to \Omega_n$ or $\Omega_n \to SK_n$.

$\overline{SK}_n :=$ n-manifolds up to cobordism <u>and</u> SK equivalence

$$
\begin{tikzcd}
 & SK_n \arrow[d, two heads] \\
\Omega_n \arrow[r, two heads] & \overline{SK}_n
\end{tikzcd}
$$

Modern maths: extend one invariant to an infinite sequence!

group G associated to some process $\rightsquigarrow$ build space X associated to the process such that $\pi_0(X) = G$ $\rightsquigarrow$ study other topological invariants of X e.g. $\pi_1(X)$

connected components

new invariants!

'categorification'

joint with Mona Merling, Laura Murray, Carmen Rovi & Julia Semikina

Aim:

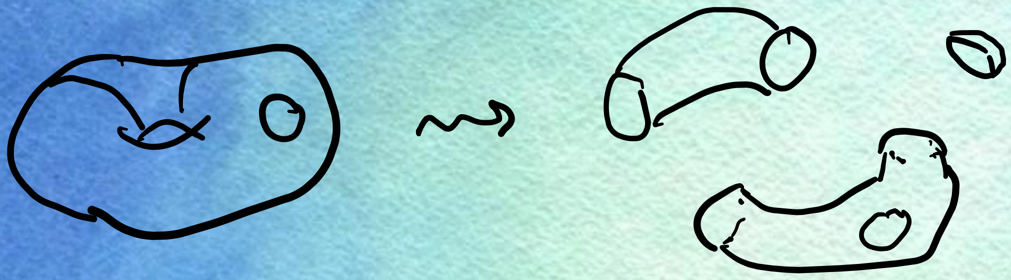
Categorify SK_n using K-theory methods similar to those built for classical scissors congruence

$\rightsquigarrow$ build a space $K^{\square}$ such that $\pi_0 K^{\square} \cong SK_n$

Problem:

cut polytope:

pieces are polytopes



cut manifold:

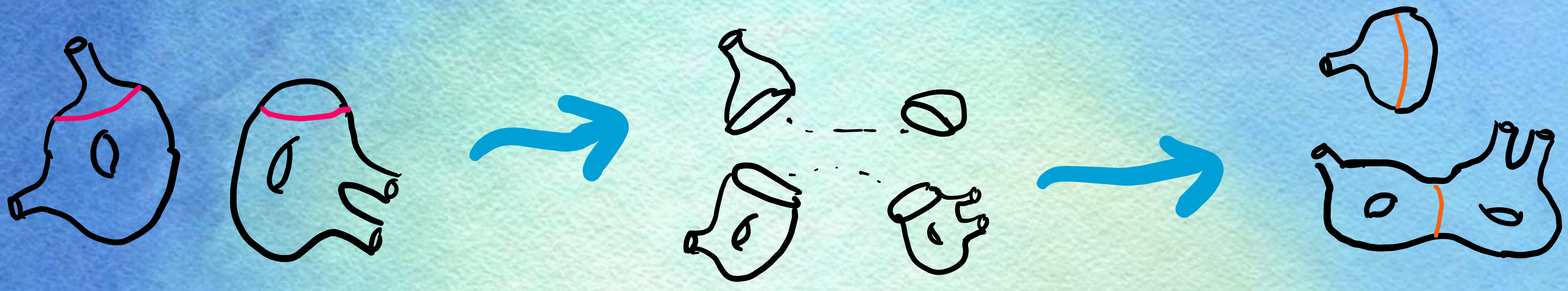
pieces are manifolds with boundary

Solution: Work instead w/ manifolds with boundary.

$$\longrightarrow \quad SK_n^{\partial}$$

Def:

$SK_n^\partial :=$ n-manifolds with ∂ up to SK^∂ equivalence



Note: Our SK^∂ equivalence preserves boundary (unlike Karras, Kreck, Neumann, Ossa)

Thm (HMMRS) There is a short exact sequence:

$$SK_n \xrightarrow{\alpha} SK_n^\partial \xrightarrow{\partial} C_{n-1}$$

group of bounding n-1 manifolds

Apply K-theory of squares, new formalism developed by Campbell & Zakharevich

<u>input</u>: "category with squares" $\mathcal{C}$ $\longrightarrow$ <u>output</u>: "K-theory space" $K^{\square}(\mathcal{C})$

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Def: A category with squares is a category $\mathcal{C}$ with distinguished squares $\square$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

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+ conditions

Space built using squares

Satisfies:

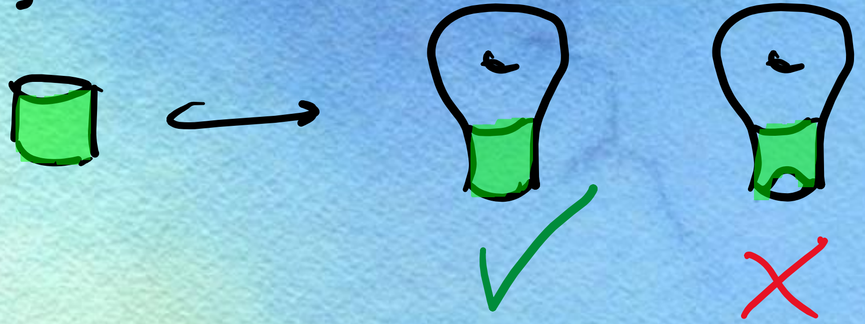
$\pi_0 K^{\square}(\mathcal{C})$ = spanned by objects of $\mathcal{C}$ up to <u>square relation</u>

$A + D = B + C$ whenever $\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \square & \downarrow \\ C & \longrightarrow & D \end{array}$

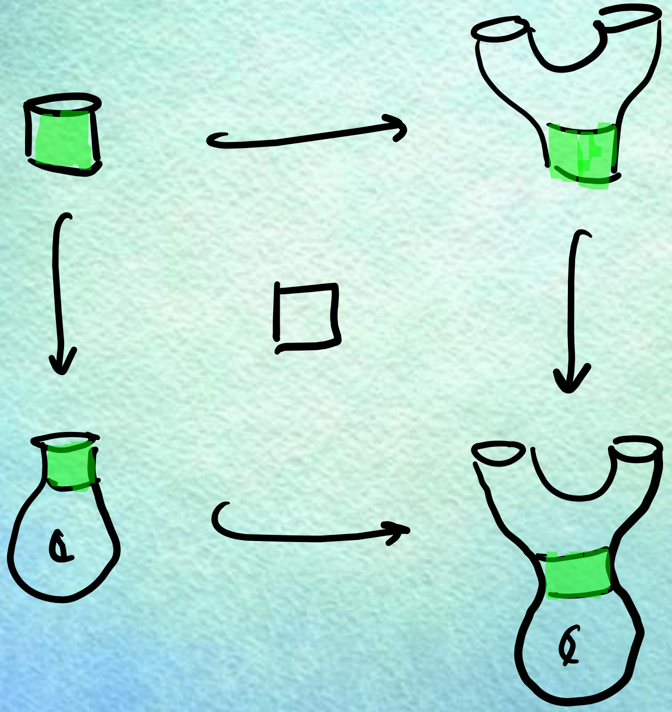
Applying K-theory of squares to manifolds:

Def: Category of manifolds with squares

ob = smooth compact n-mfds w/ ∂

mor = 

$\square$ = pushout squares



$\Rightarrow$ space $K^{\square}(\mathcal{C})$

Thm: $\pi_0 K^{\square}(\mathcal{C}) = SK_n^{\partial}$

Central idea of proof:

manifolds$^{\partial}$ up to SK^{∂} equivalence = manifolds$^{\partial}$ up to square relation

$A + D = B + C$ whenever
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

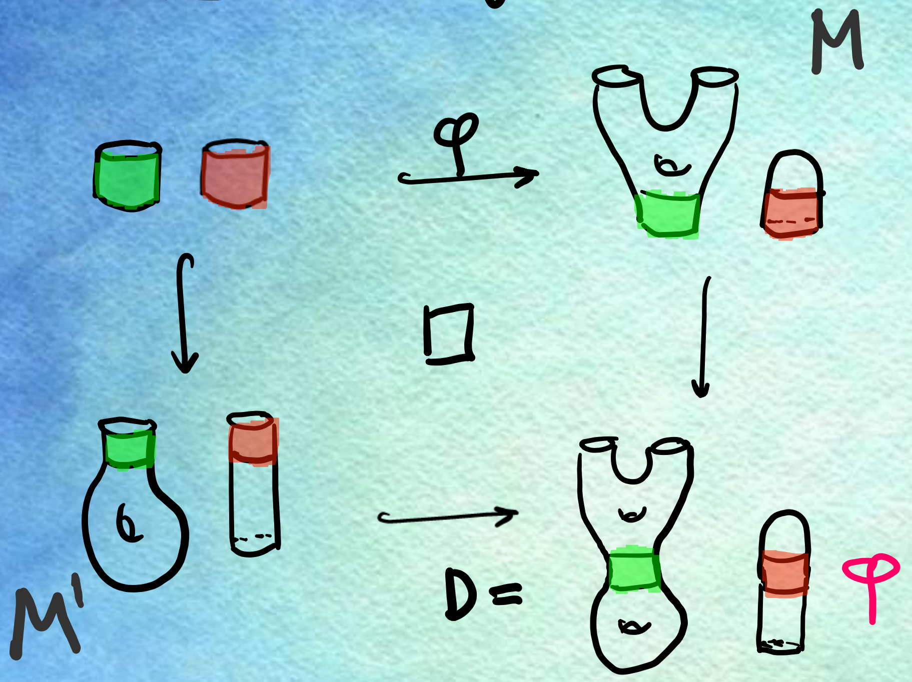
manifolds$^{\partial}$ up to SK$^{\partial}$ equivalence = manifolds$^{\partial}$ up to square relation

① square relation $\Rightarrow$ SK relation

② SK relation $\Rightarrow$ square relation

① Take SK-equivalent manifolds $D = M \cup_{\varphi} M'$ and $D' = M \cup_{\psi} M'$

Consider <u>square that glues D:</u>



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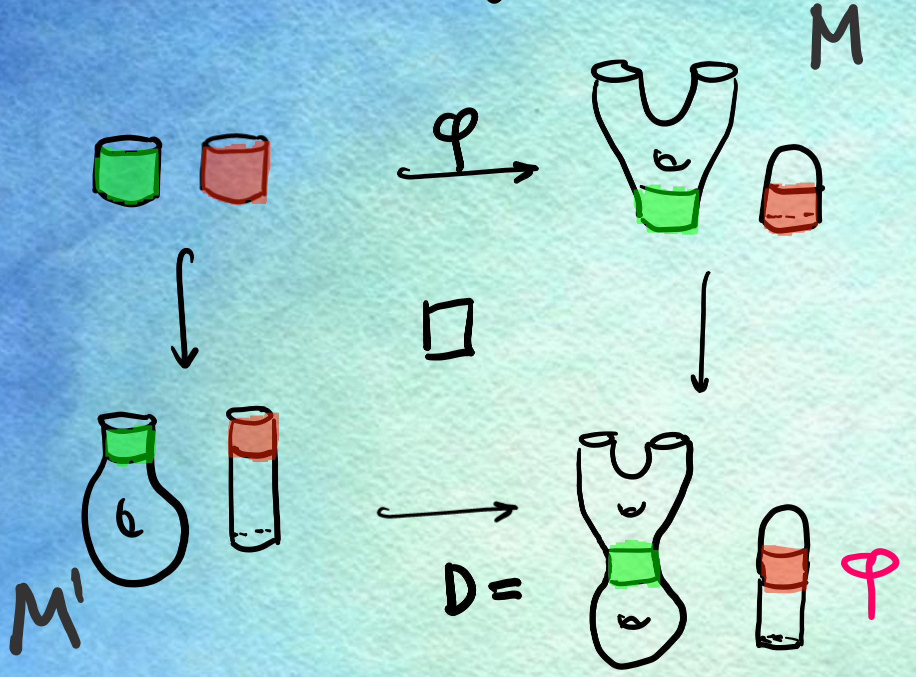

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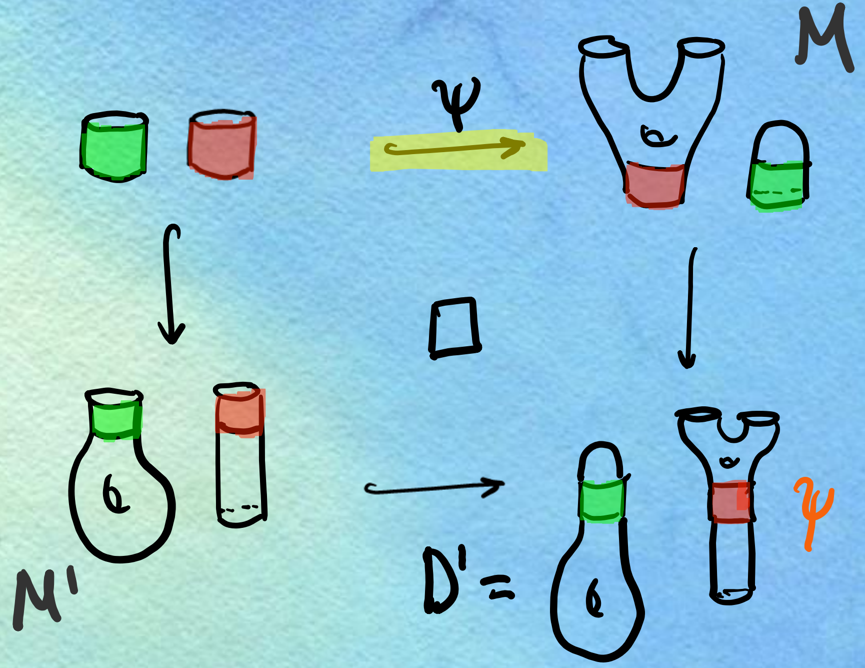

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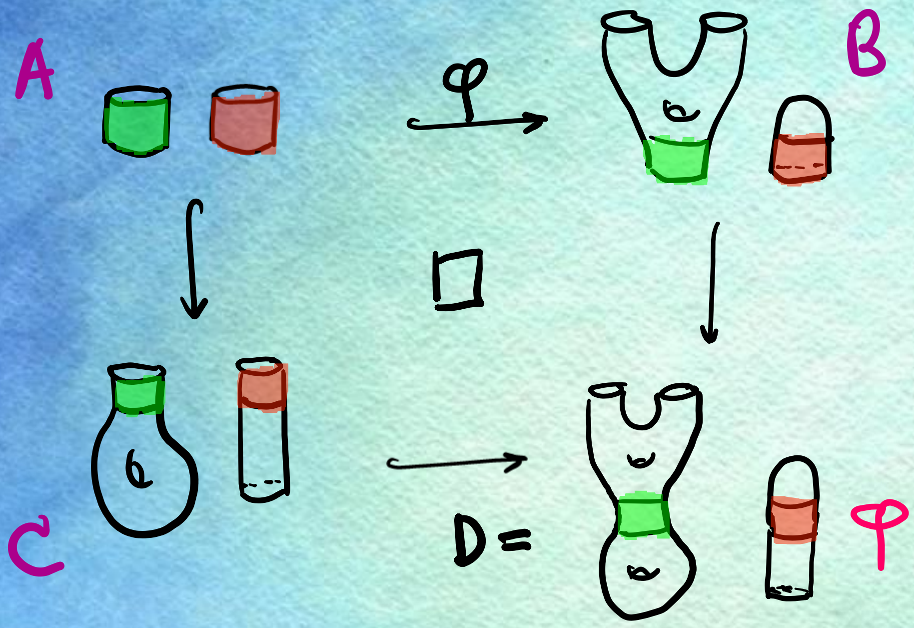

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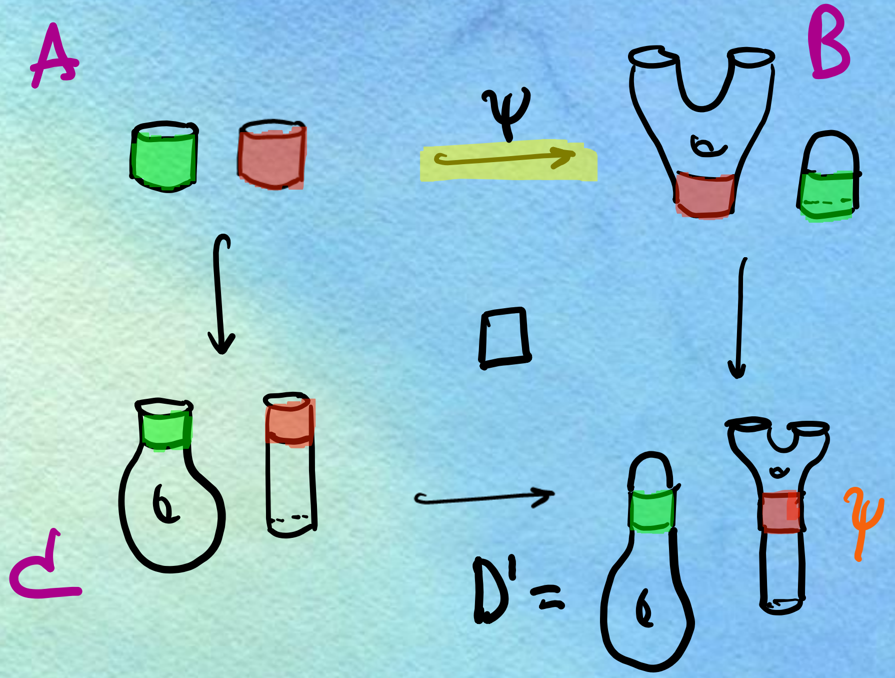


Square relation: $A + D = B + C$

$D = B + C - A$

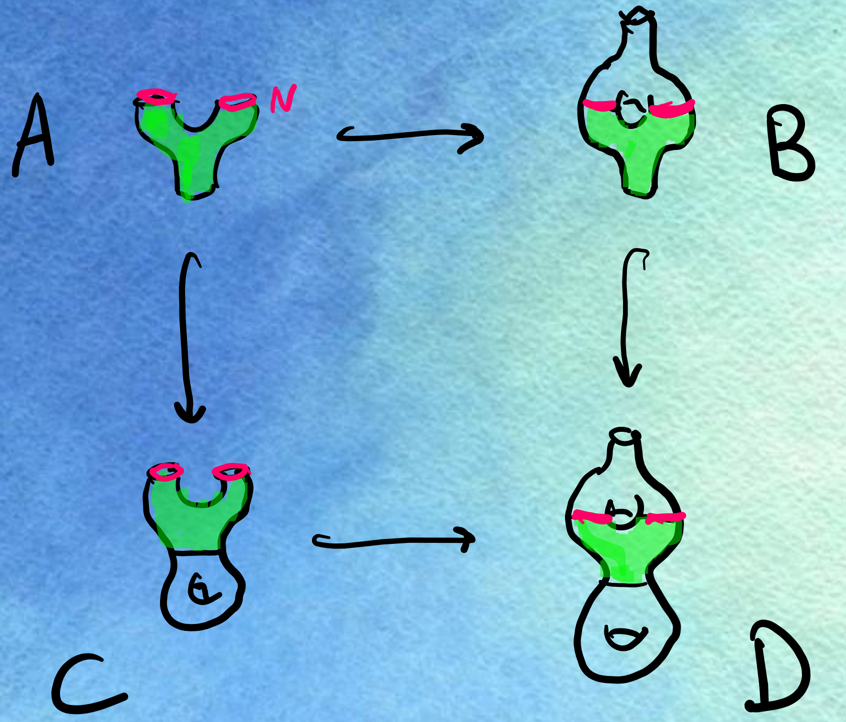
square relation: $A + D' = B + C'$

$D' = B + C' - A$

$\rightsquigarrow$ square relations imply $D = D'$

② Take a square of manifolds

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$



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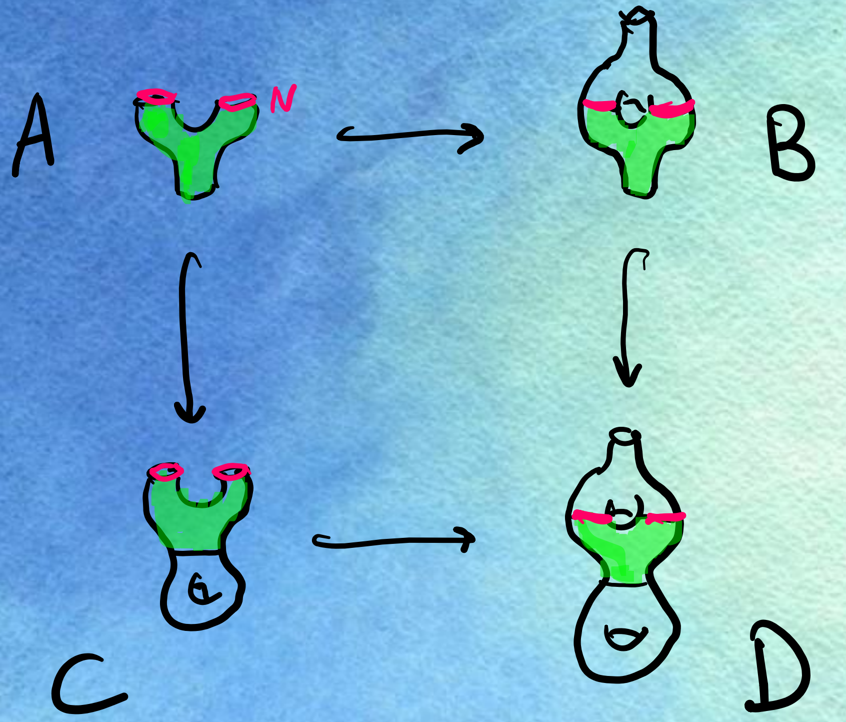

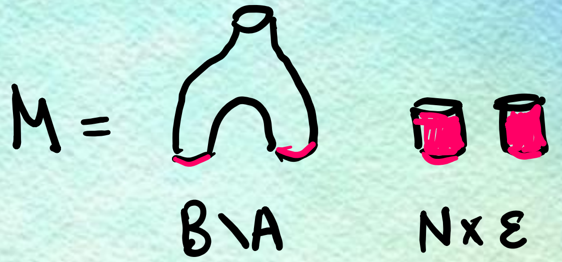

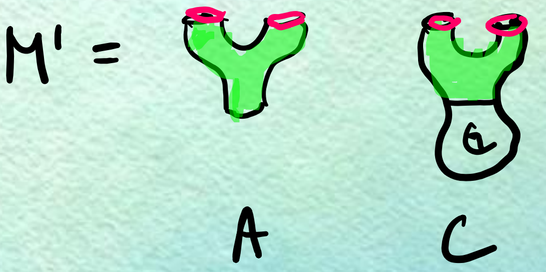

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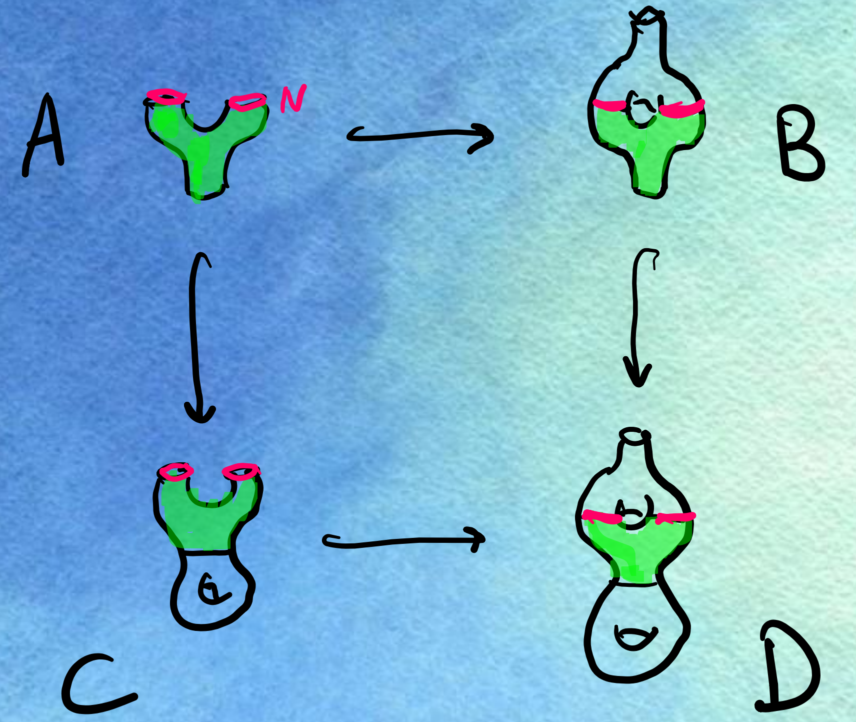
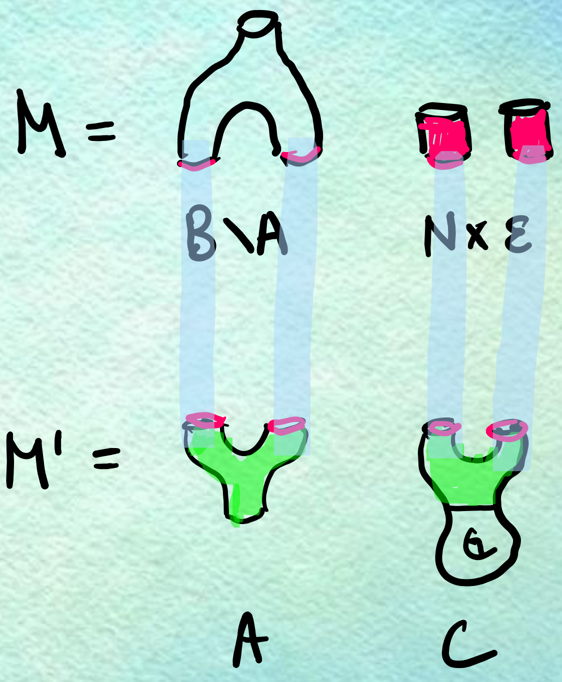
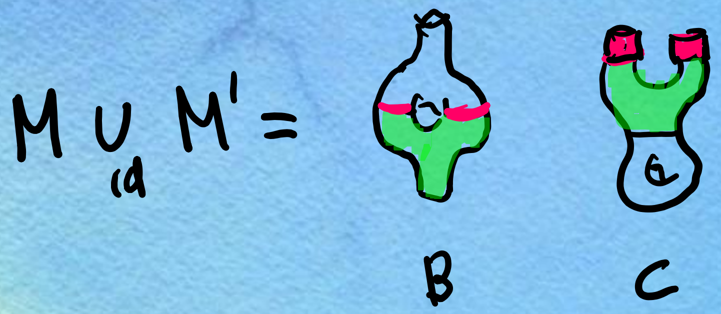
$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & D \end{array}$$







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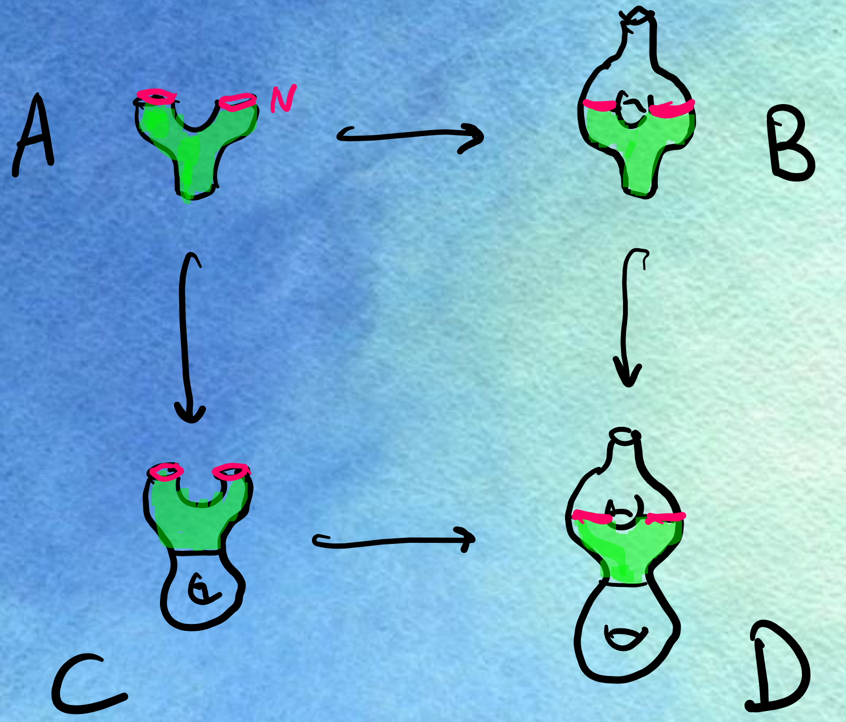

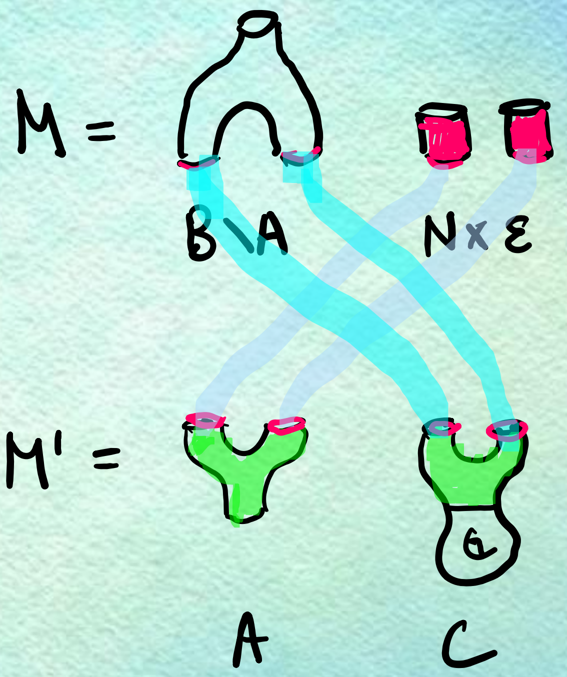

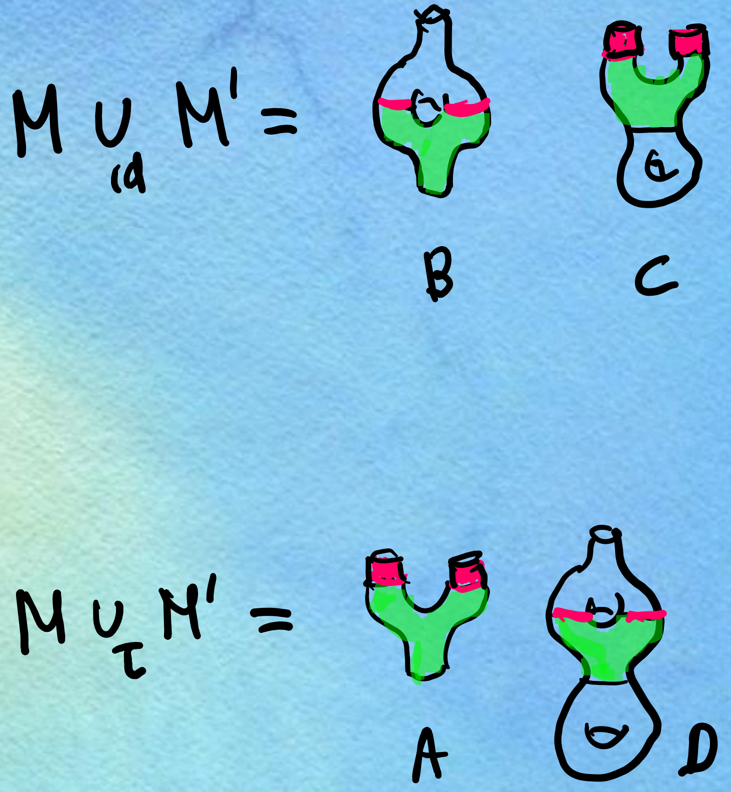

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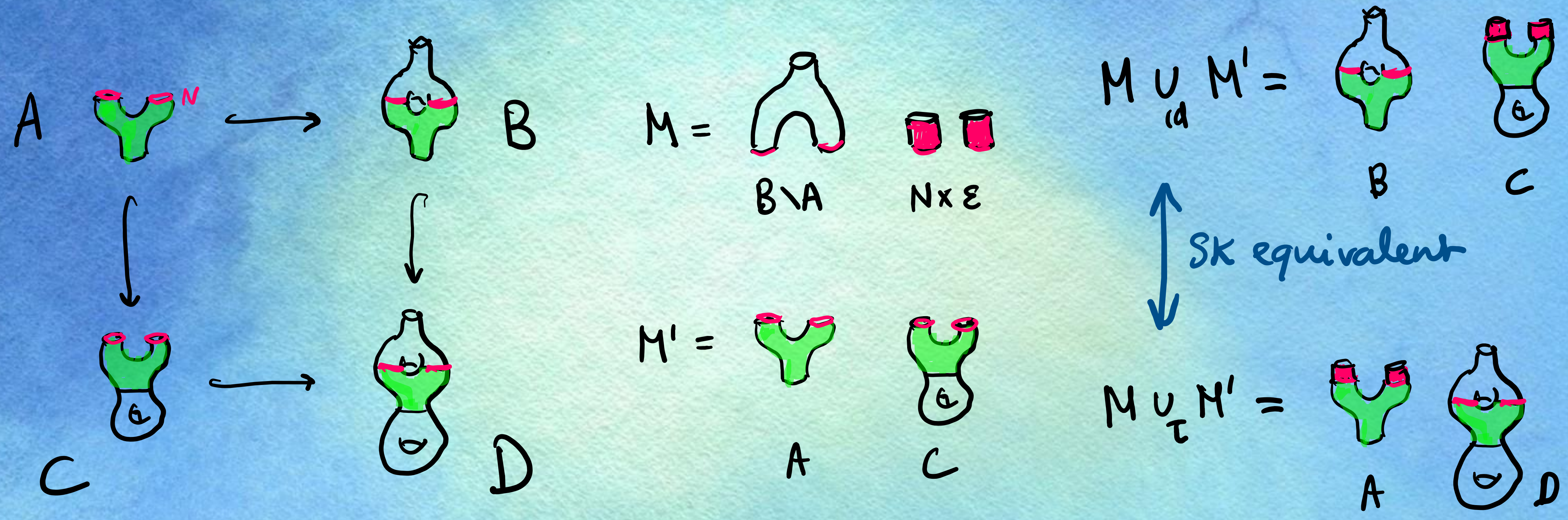
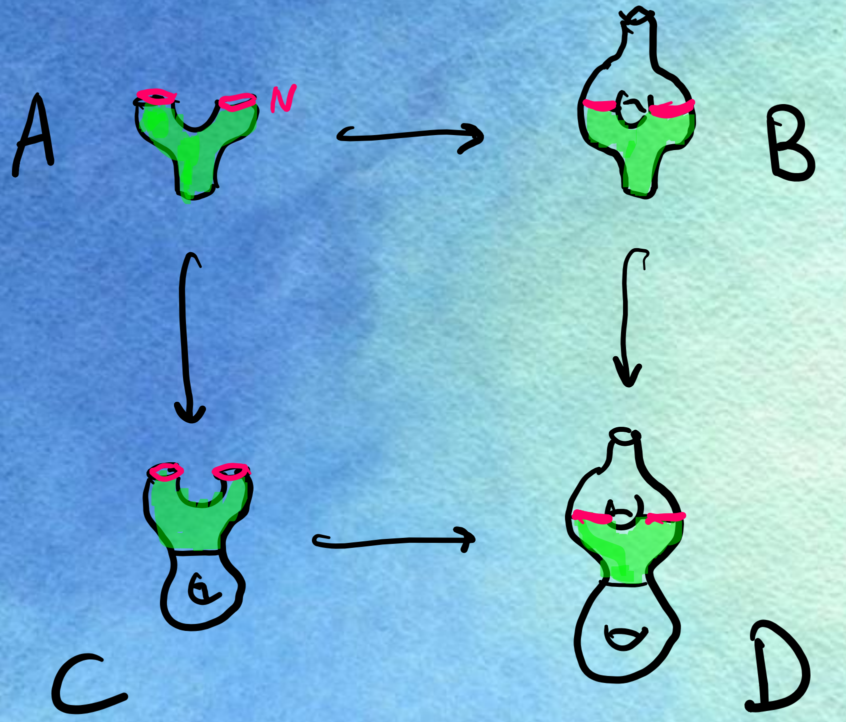
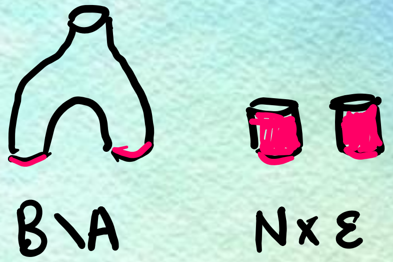


② Take a square of manifolds

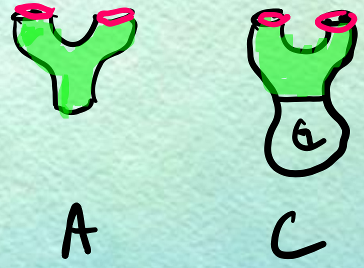
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$



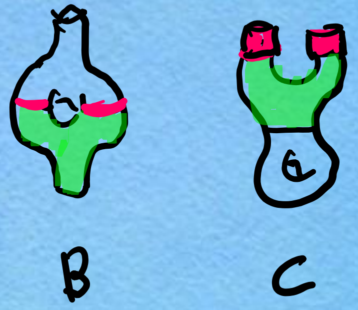
$M =$



$M' =$

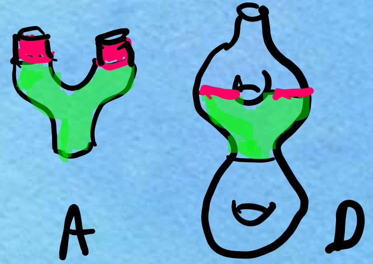


$M \cup_{id} M' =$



$\Updownarrow$ SK equivalent

$M \cup_{\tau} M' =$

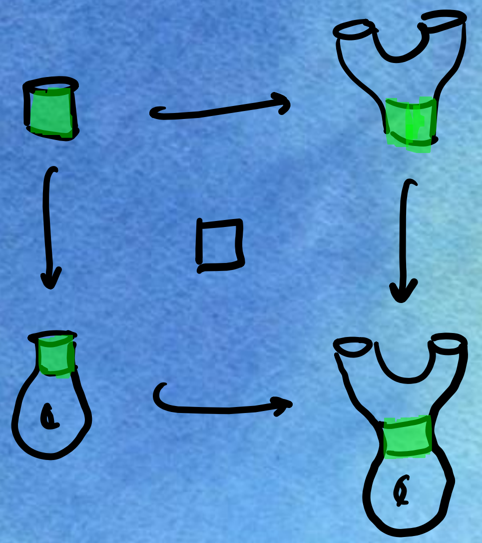


$\leadsto$ SK equivalence implies $B + C = A + D$

$\boxtimes$

So,

we've looked at SK-equivalence of manifolds



we 'categorified' SK_n^∂:

- the space $K^\square(\mathcal{C})$ has connected components SK_n^∂

SK-equivalence $\leftrightsquigarrow$ squares of manifolds

$\pi_1 K^\square(\mathcal{C}) \rightsquigarrow$? 'higher cut and paste invariants' may tell us new things about manifolds!

Lots of cutting & pasting
to be done.

Thanks!